AM-FM EXPANSIONS FOR IMAGES

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ABSTRACT

In this paper we present a novel method for computing AM-FM expansions for images. Given an image, we show how to compute an appropriate AM-FM representation. We also describe a general class of functions for which this approach gives the best results. Then, we compute the AM-FM representation on a real-life texture, and show that it has a compact AM-FM spectrum.

1 INTRODUCTION

We begin by introducing the AM-FM series expansion for images. The model seeks to describe images in terms of a sum of AM-FM components. For an image g, we write [2]:

$$g\left(x_1,\, x_2\right) = \sum_n a_n\left(x_1,\, x_2\right) \exp\left\{\, j\phi_n\left(x_1,\, x_2\right)\,\right\} \qquad (1)$$

where a_n denotes a slowly-varying amplitude function, and ϕ_n denotes a slowly-varying phase function. The model (as given in (1)) is very general, capable of describing non-stationary image features using a small number of AM-FM components [3].

For describing arbitrary images using AM-FM components, we next introduce a new AM-FM representation. For general images g, we study general classes of amplitude functions $a\left(.,.\right)$ and phase functions $\phi\left(.,.,.,.\right)$ that permit the AM-FM representation:

$$g(x_1, x_2) = \iint_{-\infty}^{+\infty} \mathbf{G}\left(f_1, f_2\right) a(x_1, x_2) \exp\left\{\, j\phi(x_1, x_2, f_1, f_2)\,\right\}\, df_1\, df_2 \qquad (2)$$

where $\mathbf{G}\left(f_1, f_2\right)$ is defined to be the AM-FM spectrum function. Using (2), we can reconstruct g from its AM-FM components. As given in (2), for reconstruction, we multiply each component by its corresponding coefficient $\mathbf{G}\left(f_1, f_2\right)$ and add the contributions from each. Thus, to compute the AM-FM transform expansions of (2), we must specify: $a(x_1, x_2)$, $\phi(x_1, x_2, f_1, f_2)$, and $\mathbf{G}(f_1, f_2)$.

The solution to the representation problem given in (2) is described in section 2. In section 2, we discuss how to compute appropriate AM-FM representations from any given image. Then, we describe a class of functions for which the AM-FM representation of (2) produces the best results. We compute the AM-FM representation for a real-life texture in section 4.

2 The AM-FM Representation Theorem

In this section, we show that AM-FM transforms (as given in (2), can be constructed in terms of positive amplitude functions: $a > 0$, and for ϕ expressed in terms of coordinate functions $\phi_1,\ \ \phi_2$. The result is summarized in the form of a theorem for ease of reference.

Consider the following simple form for the phase function:

$$\phi(x_1, x_2, f_1, f_2) = 2\pi\left(f_1\phi_1(x_1, x_2) + f_2\phi_2(x_1, x_2)\right) \quad (3)$$

where $\mathbf{\Phi}\left(x_1, x_2\right) \equiv \left(\phi_1(x_1, x_2), \phi_2(x_1, x_2)\right)$ constitutes a coordinate transformation from the $x_1 - x_2$ coordinate system to the $\phi_1 - \phi_2$ coordinate system. This means that $\mathbf{\Phi}$ is continuously - differentiable, one to one, and its jacobian is never zero over the entire plane [1]. Furthermore, we restrict our attention to positive amplitude signals $a > 0$, for which the Fourier Transform of g/a exists (g being the given image function). The following theorem says that we can always write (2) for this case.

Theorem 1 (AM-FM transform theorem) *Let $a > 0$ be given. Let g be an image function for which the Fourier transform of g/a exists. Let $\mathbf{\Phi}\left(x_1, x_2\right) \equiv \left(\phi_1(x_1, x_2), \phi_2(x_1, x_2)\right)$ be any given coordinate transformation. Then, g can be expressed as:*

$$g(x_1, x_2) = \iint_{-\infty}^{+\infty} \mathbf{G}\left(f_1, f_2\right)\, a(x_1,\, x_2) \exp\left\{\, j2\pi\left(\, f_1\,\phi_1\left(x_1, x_2\right) + f_2\,\phi_2\left(x_1, x_2\right)\right)\,\right\}\, df_1\, df_2 \quad (4)$$

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where (as before), $\mathbf{G}\,(f_1, f_2)$ *denotes the AM-FM spectrum function, which can be computed using:*

$$\mathbf{G}\,(f_1,\, f_2) \;=\; \iint_{-\infty}^{+\infty} \frac{g}{a}\,\left(\mathbf{\Phi}^{-1}\,(\phi_1,\, \phi_2)\right) \exp\{\,-j2\pi\,(f_1\phi_1 + f_2\phi_2)\,\}\, d\phi_1\, d\phi_2 \qquad (5)$$

Proof:
Since $a > 0$, we see that the function g/a is well-defined. Using the rectangular coordinate system $x_1 - x_2$ and the coordinate transformation $\mathbf{\Phi}$, we define a new function h by:

$$h\,(\mathbf{\Phi}\,(x_1,\, x_2)) \equiv \frac{g}{a}\,(x_1,\, x_2) \qquad (6)$$

By assumption, we can write the Fourier transform of g/a, and hence:

$$h\,(\mathbf{\Phi}\,(x_1,\, x_2)) = \iint_{-\infty}^{+\infty} H\,(f_1,\, f_2) \exp\{\,j2\pi\,(f_1\phi_1\,(x_1,\, x_2) + f_2\phi_2\,(x_1,\, x_2))\,\}\, df_1\, df_2 \qquad (7)$$

Furthermore, for g, we write:

$$g\,(x_1,\, x_2) = a\,(x_1,\, x_2)\; h\,(\mathbf{\Phi}\,(x_1,\, x_2)) = \iint_{-\infty}^{+\infty} H\,(f_1,\, f_2)\, a\,(x_1,\, x_2) \exp\{\,j2\pi\,(f_1\phi_1\,(x_1,\, x_2) + f_2\phi_2\,(x_1,\, x_2))\,\}\, df_1\, df_2 \qquad (8)$$

In (8), we recognize the AM-FM basis functions:

$$a\,(x_1,\, x_2)\, \exp\{\,j2\pi\,(f_1\phi_1\,(x_1,\, x_2) + f_2\phi_2\,(x_1,\, x_2))\,\}$$

weighted by the AM-FM spectrum coefficients $H\,(f_1,\, f_2)$. Thus, to complete the proof, we need only show how to compute the Fourier transform of h in terms of $a,\;\; g$. Since $\mathbf{\Phi}$ is a coordinate-transformation, its inverse $\mathbf{\Phi}^{-1}$ exists. This allows us to express h in terms of g and a by using:

$$h\,(\phi_1,\, \phi_2) \;=\; h\,\left(\mathbf{\Phi}\,\left(\mathbf{\Phi}^{-1}\,(\phi_1,\, \phi_2)\right)\right) \qquad (9)$$

$$\;=\; \frac{g}{a}\,\left(\mathbf{\Phi}^{-1}\,(\phi_1,\, \phi_2)\right) \qquad (10)$$

From (10), we immediately get the AM-FM spectrum function $\mathbf{G}\,(f_1,\, f_2)$:

$$\mathbf{G}\,(f_1,\, f_2) \;=\; H\,(f_1,\, f_2) \qquad (11)$$

$$\;=\; \iint_{-\infty}^{+\infty} \frac{g}{a}\,\left(\mathbf{\Phi}^{-1}\,(\phi_1,\, \phi_2)\right) \exp\{\,-j2\pi\,(f_1\phi_1 + f_2\phi_2)\,\}\, d\phi_1\, d\phi_2 \qquad (12)$$

Q.E.D.

From (11), it is clear that the AM-FM spectrum is the simply the Fourier spectrum of h. Clearly though, the AM-FM representation is extremely interesting whenever the Fourier Transform of h has a simple form. We will expand on this observation in the following sections.

3 Model Computation using AM-FM demodulation techniques

In this section, we show how to compute an appropriate amplitude function $a > 0$, and coordinate system $x1_r - x2_r$ from any given image function g. Our method relies on the use of an AM-FM demodulation technique (see [3], [5]). Then, we briefly describe a class of images for which the AM-FM spectra have a particularly simple form.

First, we summarize the assumptions on what an image needs to satisfy for the AM-FM demodulation algorithm to work. For some $a > 0$, we assume that our image function g satisfies the conditions of theorem (1), for slowly-varying $a(.,.)$, and slowly-varying spatial gradient $\nabla\phi$ (as in [3]). Furthermore, we assume that an average AM-FM component gets filtered through a lattice of band-pass filters (as in [2]). For computing estimates for the amplitude function: $\hat{a} > 0$, and the instantaneous frequency $\left(\hat{\phi}_{x_1},\, \hat{\phi}_{x_2}\right)$, we use an extended version of the CESA algorithm given in [4], [6].

Next, we need to compute a new coordinate transformation $x1_r$–$x2_r$ To begin with, we construct an intermediate coordinate system $s_1 - s_2$. Using the instantaneous frequency estimates, we define:

$$s_i(x_1,\, x_2) \equiv \int_0^{x_1} ||\hat{\phi}_{x_1}(\tau,\, x_2)||\, d\tau, \quad i = 1, 2 \qquad (13)$$

It is easy to show that (13) defines a coordinate transformation from the coordinate system of $x_1 - x_2$ into the coordinate system of $s_1 - s_2$ over regions where both components of the instantaneous frequency do not vanish. The reason for requiring that the instantaneous frequency vector components should not vanish is to guarantee that the jacobian of the $s_1 - s_2$ transformation does not vanish.

Now, using the coordinate system of $s_1 - s_2$, we construct the coordinate system of $x1_r - x2_r$. Here, $x1_r - x2_r$ which is a re-scaled version of $s_1 - s_2$: $xi_r = d_{1,i}s_i + d_{2,i}, \quad i = 1, 2$. From (13), it is clear that, along each coordinate axis x_i, the corresponding s_i increases with x_i. Let us fix our attention to the case where $i = 1$. For a particular x_2 image-coordinate, let $x1_min$, $x1_max$ denote the minimum and maximum x_2-image coordinates of interest. Then, $d_{1,1}$ and $d_{2,1}$ are computed so that: $s_1 = 0$ gets mapped to $x_1 = x1_min$ and $s_1(x1_max, x_2)$ gets mapped to $x1_max$. Similarly, $d_{1,1}$ and $d_{2,1}$ are computed in terms of s_2. The new coordinate system $x1_r - x2_r$ is defined in the original image plane of $x1 - x_2$, and in discrete form, it is a sampling lattice for the image g (hence the subscript r in xi_r, for re-sampling).

Next, we consider a large class of functions that have very simple AM-FM spectra. Consider images g, expressed as:

$$g(x_1, x_2) \equiv a(x_1, x_2)h(\phi_1(x_1, x_2)) \qquad (14)$$

Clearly, we would expect that, in the $\phi_1 - \phi_2$ coordinate system, and for $a > 0$, that the AM-FM spectrum $\mathbf{G}(f_1, f_2)$ is only supported along the axis $f_2 = 0$ (where f_2 corresponds to the ϕ_2 coordinate). It is easy to show that this is true, and the proof follows directly from the fact that the AM-FM spectra are unique for a particular choice of coordinate system and amplitude function. Assuming perfect estimation, it can also be shown that for an image satisfying (14), the AM-FM spectrum under the $x1_r - x2_r$ coordinates, and $a > 0$ amplitude function is supported along a line in the $f_1 - f_2$ plane. The proof follows from the definitions of the coordinate systems.

It is interesting to interpret (14) in terms of the language of the communications literature. In the communications literature, we view h as a carrier function that gets modulated by the coordinate function ϕ_1 and the amplitude function $a(.,)$. Furthermore, from the derivation of the AM-FM transform theorem (see (11)), it is clear that under the $\phi_1 - \phi_2$ coordinate system, and amplitude function $a(.,.)$, the AM-FM spectrum is simply the Fourier transform of the carrier signal h. It can be shown that even under the $x1_r - x2_r$ coordinate system, and $a(.,.)$, the AM-FM spectrum is still only a function of the carrier signal h. We will return to this point in the next section.

4 Results

In Figure 1, we show the results of computing an AM-FM expansion for a woodgrain image. The instantaneous frequency estimates were obtained using the AM-FM demodulation technique described in [5]. For simplicity, we set the amplitude function to unity: $a(.,.) = 1$. The $s_1 - s_2$ coordinates were computed by approximating (13) by using sums. From the estimated $s_1 - s_2$ coordinates, the $x1_r - x2_r$ coordinates were estimated by using inverse linear interpolation on the $s_1 - s_2$ coordinates. By inverse linear interpolation, for a particular direction on the image plane (rows or columns), we mean that the range of s_i was uniformly sampled, and inverse interpolation was used to determine the xr_i for which s_i takes these values.

By examining Figure 1 (b) closely, we see that the re-sampled woodgrain image appears to be nearly periodic under the new coordinate system of $x1_r - x2_r$. Thus, roughly speaking, the original image can be approximated by a an image of the type given in (14), where a one-dimensional periodic signal (similar to the one in Figure 1(b)), gets modulated into the one in Figure 1(a). This observation helps explain why the AM-FM spectrum of Figure 1 (f) is much more compact than the original FFT spectrum of Figure 1 (e), and also why the AM-FM spectrum appears to be spread about a line through the origin.

5 Future work

In future work, we will consider the problem of reconstructing sampled images from their AM-FM spectra. For reconstruction purposes, we are curently researching non-uniform sampling techniques. We are also researching the invariant nature of the AM-FM spectra.

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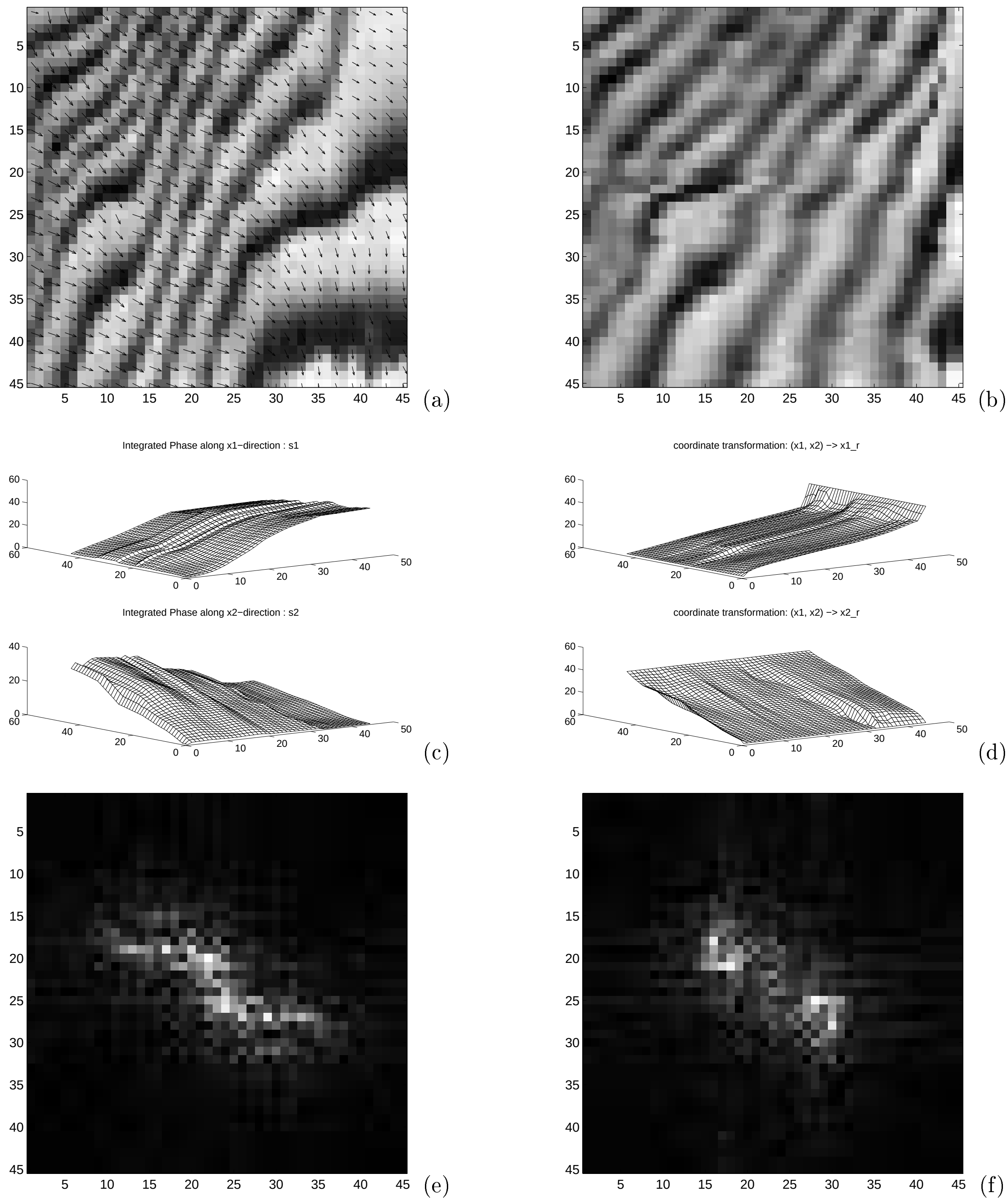

Figure 1: An AM-FM transform example on a real image. (a) The extracted wood-grain image with instantaneous frequency estimates. Every other instantaneous frequency vector is shown. Each vector magnitude M is re-scaled using $\log(1 + 100 * M)$. (b) The original image re-sampled in the $x1_r - x2_r$ coordinate system. (c) The coordinate transformation functions s_1 (x_1, x_2) (upper plot), and s_2 (x_1, x_2) (lower plot). (d) The coordinate transformation functions $x1_r$ (x_1, x_2) (upper plot), and $x2_r$ (x_1, x_2) (lower plot). (e) The magnitude plot of the FFT spectrum of (a). (f) The magnitude plot of the AM-FM spectrum of (a), (or the magnitude FFT spectrum of (b)).