LEAST SQUARES FIR FILTER DESIGN USING FREQUENCY DOMAIN PIECEWISE POLYNOMIAL APPROXIMATIONS

Marios S. Pattichis
Dept. of EECE and High Performance Computing Center
The University of New Mexico, Albuquerque, New Mexico 87131, USA
Tel: (505) 277-2436; fax: (505) 277-1439
e-mail: pattichis@eece.unm.edu

Abstract-- A general framework for specifying and designing one-dimensional FIR digital filters is presented. The system uses piecewise polynomials to guarantee that the ideal impulse response maintains continuity up to a given number of derivatives (given as parameter p). In this system, it is possible to efficiently design optimal (in the least-squares sense) digital filters that converge pointwise to any (continuous) desired frequency response, while the filters' coefficients decay in the order of $1/n^{p+1}$. It is shown experimentally that the maximum absolute error also decays as $1/n^{p+1}$. Results on (i) bandpass filter design and (ii) bandpass differentiating filter design are shown.
Index Terms-- digital filter design, piecewise-polynomials.

I. INTRODUCTION

In this paper, a new method for computing digital filters is presented. The new method uses frequency domain piecewise polynomials to approximate an ideal frequency response.

The use of frequency domain polynomials to design digital filters is best known in the Parks-McClellan Algorithm for Optimal FIR filter design. In the Parks-Mclellan Algorithm, the frequency response of the digital filter to be designed is mapped to a frequency domain polynomial using the substitution: $x = \cos(w)$. The algorithm computes the optimum digital filter coefficients (optimum in the min-max sense) by computing the unique frequency domain polynomial that achieves the maximum absolute deviation from the ideal frequency response at a number of alteration frequencies (which are also estimated) [1, 2].

The use of frequency domain, piecewise polynomials is also in common practice in computing least squares approximations to ideal frequency response specifications of digital filters. However, in such least-squares designs, the ideal frequence response specifications exhibit discontinuities. These discontinuites prevent the digital filter approximations from achieving pointwise convergence to the ideal frequency response [2]. A variation of this method allows the specification of specific functions for the transition regions (Section 3.2 in [1]). The approach presented here is a generalization of the transition region approach.

Let the desired frequency response be specified in terms of a collection of piecewise polynomials:

$$D\left(e^{jw}\right) = \begin{cases} p_1(w) & \text{for } w \in (-p, w_1) \\ \vdots & \vdots \\ p_m(w) & \text{for } w \in (w_m, p) \end{cases} \quad (1)$$

In terms of the desired frequency coefficients, the digital filter coefficients are given by:

$$h_d[n] = \frac{1}{2p}\int_{-p}^{p} D\left(e^{jw}\right) e^{jwn}\, dw \quad (2).$$

Using the method of jumps (see [3]), it is possible to compute $h_d[n]$ in terms of the jumps in the derivatives of the polynomial pieces. Let us define $j_s^{(q)}$ by: $j_1^{(q)} = p_1^{(q)}(-p^-) - p_m^{(q)}(p^+)$, and $j_s^{(q)} = p_s^{(q)}(w_s^+) - p_{s-1}^{(q)}(w_s^-)$ for $s = 2, 3, \ldots, m$ (where w_s^+ denotes a right hand limit, while w_s^- denotes a left hand limit at w_s). The digital filter coefficients are then given by:

$$h_d[0] = 1/(2p)\int_{-p}^{p} D\left(e^{jw}\right) dw\,,$$

$$h_d[n] = (a_n + jb_n)/2\,, \text{ for } n > 0\,, \quad h_d[n] = h_d^*[-n]$$

for $n < 0$, where:

$$\begin{aligned} a_n \quad = \quad & \frac{1}{np}\Big[-\sum_{s=1}^{m} j_s^{(0)} \sin(nw_s) - \frac{1}{n}\sum_{s=1}^{m} j_s^{(1)} \cos(nw_s) \\ & + \frac{1}{n^2}\sum_{s=1}^{m} j_s^{(2)} \sin(nw_s) + \frac{1}{n^3}\sum_{s=1}^{m} j_s^{(3)} \cos(nw_s) \\ & - - + + \cdots \Big], \quad (3) \end{aligned}$$

$$b_n = \frac{1}{np}\Big[\sum_{s=1}^{m} j_s^{(0)}\cos(nw_s) - \frac{1}{n}\sum_{s=1}^{m} j_s^{(1)}\sin(nw_s) - \frac{1}{n^2}\sum_{s=1}^{m} j_s^{(2)}\cos(nw_s) + \frac{1}{n^3}\sum_{s=1}^{m} j_s^{(3)}\sin(nw_s) + - - + + \cdots\Big]. \quad (4)$$

For least squares error design, we compute $h_d[n]$, $n = -L, \ldots, +L$ using (3) and (4). In order to guarantee pointwise convergence, we must select $p_1, \ldots, p_s$ such that $D(e^{jw})$ is continuous. Furthermore, to guarantee that the filter coefficients decay in the order of $1/n^{m+1}$, we must select $p_1, \ldots, p_s$ such that $D(e^{jw})$ is continuous up to the mth derivative. Due to the decay in the filter coefficients, and the fact that the partial sums defined by the filter coefficients can be shown to form an alternating series, easily computable upper and lower bounds for the maximum absolute error from the ideal response can be determined [6]. An experimental verification of this assertion is given in Figure 5.

In Section III, we shall see how the digital filter design problem can be solved by computing each polynomial piece independently of the others. This allows us to investigate different designs without having to solve a global optimization problem. In addition, using (3) and (4), it is clear that the transition bands can be adjusted, and the digital filter coefficients can be recomputed in real time. Also, since there is sufficient polynomial decay in the filter coefficients, we can derive min-max error bounds for the deviation from the desired frequency response (see [5] for some general convergence results). Perhaps the greatest advantage of this approach is that the method should be extensible to address the problem of designing digital filters of finite dimensions. In fact, the initial problem of computing polynomials that meet different criteria in two and three dimensions is very well understood in Computer Graphics (see for example [4]). It is natural to expect that the same degree of success that piecewise polynomials have enjoyed in Computer Graphics should be extensible to the problems of multidimensional digital filter design.

The general algorithm for computing a single polynomial p_i is presented in Section II. Using the algorithm of Section II, a bandpass filter design example is presented in Section III. A bandpass differentiator is designed in Section IV. Some concluding remarks are summarized in Section V.

II. On the Specification and Computation of a Single Polynomial

In this Section, we address the problem of computing a single polynomial $p_i(w)$. The method is similar to the Parks-McClellan algorithm in that the polynomial coefficients are computed so that the polynomial satisfies certain conditions at its endpoints.

To compute $p_i(w)$, we specify the polynomial derivatives at its boundaries w_i and w_{i+1}:

$$\mathbf{b} = \left[p_i(w_i), p_i(w_{i+1}), \ \ldots, p_i^{(m)}(w_i), p_i^{(m)}(w_{i+1})\right]^T,$$

and then compute the polynomial coefficients:

$$\mathbf{y} = [a_0, a_1, \ldots, a_{2m+1}]^T$$

of $p_i(w) = a_0 + a_1 w + \cdots + a_{2n+1} w^{2m+1}$. Using $\mathbf{b}$ and $\mathbf{y}$, we have $\mathbf{Ay} = \mathbf{b}$ where the coefficient matrix $\mathbf{A}$ is (using general expressions to illustrate the pattern):

$$\begin{bmatrix}
1 & w_i & w_i^2 & \ldots & \ldots & w_i^{2m+1} \\
1 & w_{i+1} & w_{i+1}^2 & \ldots & \ldots & w_{i+1}^{2m+1} \\
 & 1 & 2w_i & 3w_1 & \ldots & (2m+1)w_1^{2m} \\
 & 1 & 2w_{i+1} & 3w_{i+1} & \ldots & (2m+1)w_{i+1}^{2m} \\
 & & \vdots & \vdots & & \\
 & & & \binom{m}{m} w_i^0 & \ldots & \binom{2m+1}{m} w_i^{m+1} \\
 & & & \binom{m}{m} w_{i+1}^0 & \ldots & \binom{2m+1}{m} w_{i+1}^{m+1}
\end{bmatrix}$$

Hence, to compute $p_i(w)$, we simply solve the linear system of equations: $\mathbf{Ay} = \mathbf{b}$.

III. Bandpass Filter Design Example

In this Section, we use frequency domain piecewise polynomials to design bandpass filters. Simulation results for different filter orders are presented.

Due to the frequency domain symmetry in the specification of bandpass filters (see Figure 1), we do not need to compute the piecewise polynomials for negative frequencies. Using this observation we save half the amount of computation time. We use the freqeuncy response given by (for non-causal FIR coefficients):

$$D(e^{jw}) = \begin{cases} 0, & \text{for } w \in (-p, w_1), \\ p_1(w), & \text{for } w \in (w_1, w_2), \\ 1, & \text{for } w \in (w_2, w_3), \\ p_2(w), & \text{for } w \in (w_3, w_4), \\ 0, & \text{for } w \in (w_4, p). \end{cases}$$

Then, after computing the digital filter coefficients for $D\left(e^{jw}\right)$, the coefficients for the final desired frequency response $D\left(e^{jw}\right)+D\left(e^{-jw}\right)$ are given by $h_e[n]=a_n$ where a_n are computed using (3).

The transition polynomials p_1 and p_2 are computed using the method described in Section II. We compute p_1 such that $p_1(w)\neq 0$ for $w\in(w_1,w_2)$, and $\mathbf{b}=[0,1,0,\ \ldots,0,0]^T$ ($2m+2$ components). Similarly, we compute the transition polynomial p_2 such that $p_2(w)\neq 0$ for $w\in(w_3,w_4)$, and $\mathbf{b}=[1,0,0,\ \ldots,0,0]^T$ ($2n+2$ components).

This technique has been applied in the design of bandpass filters that approximate the desired frequency response illustrated in Figure 1. The results are shown in Figures 1, 2, and 3. As shown in Figures 2 and 3, the oscillations remain bounded and decay to zero.

IV. Bandpass Differentiating Filter Design Example

In this Section, we use frequency domain piecewise polynomials to design a bandpass differentiating filter. Simulation results for the maximum absolute error for different numbers of filter coefficients are also presented.

We use the freqeuncy response given by (for non-causal FIR coefficients):

$$D\left(e^{jw}\right)=\begin{cases}0, & \text{for } w\in(-p,w_1),\\ p_1(w), & \text{for } w\in(w_1,w_2),\\ jw, & \text{for } w\in(w_2,w_3),\\ p_2(w), & \text{for } w\in(w_3,w_4),\\ 0, & \text{for } w\in(w_4,p).\end{cases}$$

Then, after computing the digital filter coefficients for $D\left(e^{jw}\right)$, the coefficients for the final desired frequency response are computed using (3) and (4).

The transition polynomials p_1 and p_2 are computed using the method described in Section II. We compute p_1 such that $p_1(w)\neq 0$ for $w\in(w_1,w_2)$, and $\mathbf{b}=j[0,w_2,0,1,\ldots,0,0]^T$ ($2m+2$ components). Similarly, we compute the transition polynomial p_2 such that $p_2(w)\neq 0$ for $w\in(w_3,w_4)$, and $\mathbf{b}=j[w_3,0,1,0,\ldots,0,0]^T$ ($2n+2$ components).

This technique has been applied in the design of bandpass differentiating filters that approximate the desired frequency response illustrated in Figure 4. The results are shown in Figures 4 and 5. The maximum absolute error is shown to decay at a rate that is inversely proportional to the square of the number of coefficients.

V. Conclusion and Future Work

A general system for designing digital filters has been presented. Current work is focused on the development of analytic lower and upper bounds on the maximum absolute error, as a function of the number of coefficients. This will allow the determination of the minimum number of filter coefficients required for achieving a maximum absolute error [6]. Future work will also focus on extending this approach to multidimensional digital filter design.

VI. References

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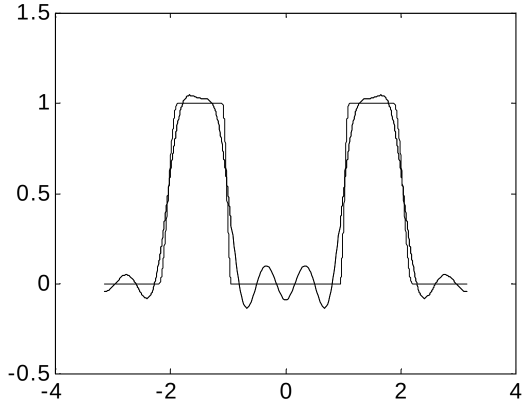

Figure 1. In this bandpass filter design, $w_1 = 0.3p$, $w_2 = 0.35p$, $w_3 = 0.6p$, $w_4 = 0.7p$. For continuity up to the first derivative, cubic polynomials are used for the transition bands. In this approximation, 19 coefficients are used (coming from 10 computed coefficients a_n for $0 \le n \le 9$).

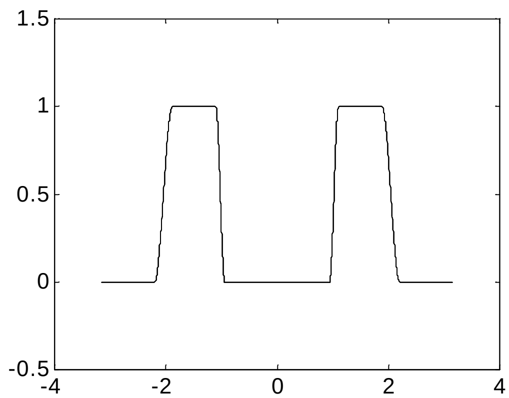

Figure 2. This is the same bandpass filter design problem as in Figure 1, but with 79 filter coefficients.

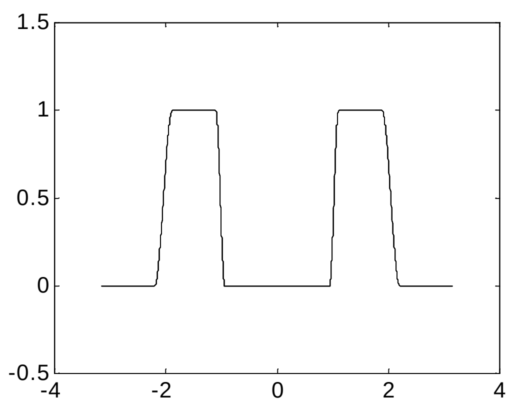

Figure 3. In this Figure, 1999 filter coefficients are used. This Figure illustrates pointwise convergence.

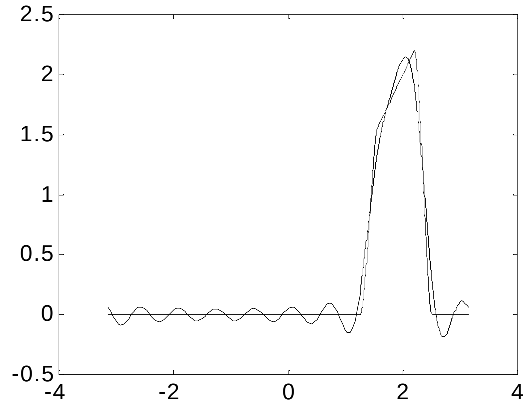

Figure 4. In this single-sideband differentiating filter design, $w_1 = 0.4p$, $w_2 = 0.5p$, $w_3 = 0.7p$ and $w_4 = 0.8p$. Cubic polynomials are used for the transition bands. In this approximation, 10 coefficients were used.

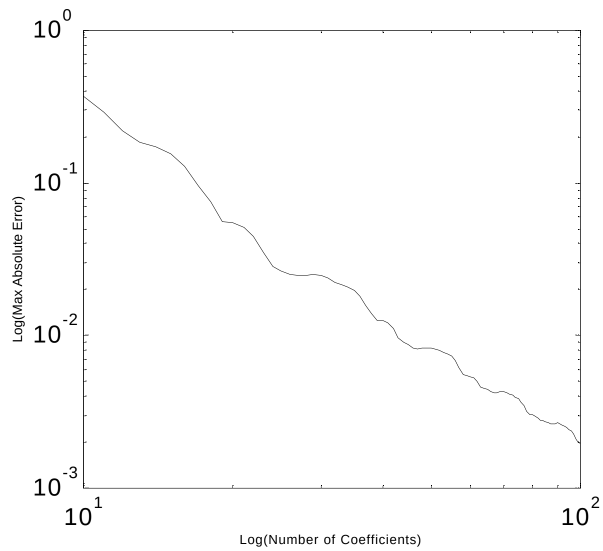

Figure 5. For the filter design problem of Figure 4, the logarithm of the maximum absolute error against the number of coefficients is shown. The maximum absolute error decays inversely proportional to the square of the number of filter coefficients.